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NNLO unquenched calculation of the b quark mass

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Abstract: By combining the first unquenched lattice computation of the B-meson binding energy and the two-loop contribution to the lattice HQET residual mass, we determine the $\overline{MS}$ b-quark mass, $\overline{m}_b(\overline{m}_b)$. The inclusion of the two-loop corrections is essential to extract $\overline{m}_b(\overline{m}_b)$ with a precision of $\mathcal{O}(\Lambda^2_{\rm QCD}/m_b)$, which is the uncertainty due to the renormalon singularities in the perturbative series of the residual mass. Our best estimate is $\overline{m}_b(\overline{m}_b) = (4.26 \pm 0.09)\,\mathrm{GeV}$, where we have combined the different errors in quadrature. A detailed discussion of the systematic errors contributing to the final number is presented. Our results have been obtained on a sample of 60 lattices of size $24^3 \times 40$ at $\beta = 5.6$, using the Wilson action for light quarks and the lattice HQET for the b quark, at two values of the sea quark masses. The quark propagators have been computed using the unquenched links generated by the TχL collaboration.

Keywords: Lattice Quantum Field Theory, Lattice Gauge Field Theories, Quark Masses and SM Parameters, Heavy Quarks Physics.

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Contents

1. Introduction

Quark masses are fundamental parameters of QCD that cannot be determined by theoretical considerations only and, due to the confinement of quarks inside hadrons, cannot be directly measured. Quark masses can, however, be introduced as short-distance effective couplings. As such, they are scale and scheme dependent quantities, the values of which depend on the adopted definition. Nonetheless, quark masses are very important for phenomenology since they enter many theoretical predictions of physical quantities such as CKM matrix elements, b-hadron inclusive semileptonic decays, total widths, etc. This is the reason why, in the last years, much effort has been devoted to accurately determine their values.

It is useful to classify quarks into two classes: light quarks, the masses of which are lower or of the order of $\Lambda_{\rm QCD}$ (the u, d and s quarks are light), and heavy quarks, with masses larger than $\Lambda_{\rm QCD}$ (b and t quarks are heavy and, to some extent, the charm quark c too). Light-quark masses are extracted from hadron spectroscopy

JHEP03(2000)018

using lattice QCD simulations [1] and also QCD sum rules and τ decay data [2]. Heavy-quark masses can be extracted from the properties of hadrons containing heavy quarks: the B-meson spectrum from the lattice HQET [3, 4], the Υ (or J/ψ) spectrum with lattice NRQCD [5, 6] or QCD Sum Rules [7]–[14] and mass effects in 3-jets $b\bar{b}g$ events [15, 16].

In this paper, we present the first unquenched HQET lattice calculation of the b quark mass. The idea [3] (see also [17]) is to combine the HQET unquenched lattice computation of the B-meson binding energy [18] with the recent next-to-next-to-leading (NNLO) perturbative calculation of the matching of the $\overline{MS}$ quark mass to its lattice HQET counterpart [4]. We stress that both unquenched simulations and NNLO matching are necessary ingredients to improve the accuracy of previous results [3]. The former is necessary to control potentially large vacuum-polarization contributions to the B-meson propagator. The latter is crucial to reduce renormalon ambiguities in the continuum-lattice matching [4]. After a careful analysis of the systematics errors, our best result is

$$\overline{m}_b(\overline{m}_b) = (4.26 \pm 0.03 \pm 0.05 \pm 0.07)\,\mathrm{GeV}\,,$$

where the first error is statistical; the second is the systematic error from the spread of values due to the use of different time intervals, fitting methods, smearing types and cube sizes for the interpolating operators, the dependence of the results on the mass of the sea quarks, the calibration of the lattice spacing and an evaluation of the $1/m_b$ corrections; the third is an estimate of the error due to the uncertainties in the values of α_s and to the effects of higher-order terms in eq. (2.13). A detailed discussion of the different errors can be found below.

The paper is organized as follows: in section 2, we briefly describe our method and give the main formulae we used; in section 3, we discuss the lattice computation of the binding energy; details of the simulation and numerical results are presented in section 4, where we also discuss the procedure used for analyzing the unquenched lattice data; in section 5, we carefully study the different sources of systematic errors in our results. Finally, in section 6, we present our final numbers and compare them with other recent determinations.

2. The method

The key idea to determine the b-quark mass consists in matching the propagator in QCD to its lattice HQET counterpart [3]. As shown below, this matching allows us to relate the pole mass to the binding energy and to the physical mass of the B meson. The renormalized $\overline{MS}$ b-quark mass at a given scale μ can then be obtained from the pole mass by using perturbation theory. In this section we briefly recall the formulae relevant to our study.

JHEP03(2000)018

Lattice HQET is an effective theory of QCD. The relation between the inverse b-quark propagator in QCD, S^{-1}, and its lattice HQET counterpart, S_L^{-1}, can be written, to lowest order in $1/m_b$, as

$$\left(\frac{1+\not{v}}{2}\right) S_P^{-1}(p, m_b; \mu) \equiv \left(\frac{1+\not{v}}{2}\right) S^{-1} \left(\frac{1+\not{v}}{2}\right) = C(\mu a, \alpha_s) \left(\frac{1+\not{v}}{2}\right) S_L^{-1}((v\cdot k)a)\,, \tag{2.1}$$

where S_P^{-1} is the projected b-quark propagator, μ the renormalization point, a the lattice spacing, $p = m_b\, v + k$ the momentum of the b-quark, v its velocity and k the residual momentum. $C(\mu\, a, \alpha_s)$ is the relevant Wilson coefficient. It contains all the mass dependence of the right hand side of eq. (2.1) since, by construction, the HQET propagator is independent of the b quark mass. It should be noticed that in order for the HQET to be applicable, k must satisfy the condition $|k| \ll m_b$. In writing eq. (2.1) we have chosen as expansion parameter the quark mass appearing in the original propagator, namely m_b.

The procedure to find C is well known: calculate the b-quark propagator in QCD and in the lattice HQET to a given order in α_s, expand the former in inverse powers of m_b to a given order (lowest order in our case), and finally compare both expressions at a fixed scale μ (with $\mu \gg \Lambda_{\rm QCD}$) to extract $C(\mu\, a, \alpha_s)$. Renormalization group can then be used to evolve this function to any scale.

To illustrate and clarify the key points of this strategy, we briefly sketch the derivation of our master formula for the b-quark mass to $O(\alpha_s)$ and then we extend our equation to include higher orders in perturbation theory. The inverse quark propagator in QCD can be written in the form

$$i\, S^{-1}(p, m_b; \mu) = \not{p} - m_b + \Sigma_1(p^2, m_b) + (\not{p} - m_b)\, \Sigma_2(p^2, m_b)\,. \tag{2.2}$$

It is very easy to calculate the self-energy form factors Σ_1 and Σ_2 to one loop in some renormalization scheme and for a fixed gauge. By writing the b-quark momentum in the B meson, p, as $p = m_b\, v + k$ and expanding in powers of $1/m_b$, one finds

$$S_P^{-1}(p, m_b; \mu) = m_b - m_b^{\rm pole} + (v\cdot k)\left[1 + \alpha_s(\mu)\left(\gamma_{D\cdot k} \ln\left(\frac{\mu}{-2(v\cdot k)}\right) + c_2\right) + \cdots\right], \tag{2.3}$$

where c_2 is a scheme dependent constant (the expression of which is irrelevant for our discussion) and γ_m and $\gamma_{D\cdot k}$ are the scheme-independent one-loop anomalous dimensions of the mass and operator $D\cdot k$, respectively.

The pole mass, $m_b^{\rm pole}$, is defined as the position of the pole of the propagator S^{-1}, at a given order in perturbation theory,

$$S_P^{-1}(\not{p} = m_b^{\rm pole}, m_b; \mu) = 0\,. \tag{2.4}$$

To one loop, the explicit calculation of eq. (2.3) gives

$$m_b^{\rm pole} = m_b\left[1 + \alpha_s(\mu)\left(\gamma_m \ln\left(\frac{m_b}{\mu}\right) + c_1\right)\right]. \tag{2.5}$$

In order to implement the matching S_L^{-1}, the propagator of the lattice HQET, must be evaluated at the same order in perturbation theory

$$S_L^{-1}((v{\cdot}k)a) = (v{\cdot}k)\left[1+\alpha_s(a)\left(\gamma_{D\cdot k}\ln\left(\frac{1}{-2(v\cdot k)a}\right)+d_2\right)\right]-\alpha_s(a)\frac{X_0}{a}+\cdots \quad (2.6)$$

where d_2 is a scheme dependent constant. From eq. (2.6) we learn that an additive, linearly divergent mass term is generated on the lattice: the so-called residual mass, δm. Inserting eq. (2.6) into eq. (2.3) and taking into account the expression of the pole mass in eq. (2.5), we obtain

$$S_P^{-1}(p,m_b;\mu) = m_b - m_b^{\rm pole} + \alpha_s(\mu)\,\frac{X_0}{a} + C(\mu\, a, \alpha_s)\, S_L^{-1}((v\cdot k)a)\,, \quad (2.7)$$

where the Wilson coefficient has the form

$$C(\mu\, a, \alpha_s) = 1 + \alpha_s(\mu)\,(\gamma_{D\cdot k}\,\ln(\mu\, a) + c_2 - d_2) \quad (2.8)$$

and we have used the fact that the difference between $\alpha_s(\mu)$ and $\alpha_s(a)$ is $\mathcal{O}(\alpha_s^2)$. Comparing eq. (2.1) with eq. (2.7), the important relation between the HQET expansion mass parameter m_b and the pole b-quark mass can be derived

$$m_b^{\rm pole} = m_b + \alpha_s(\mu)\,\frac{X_0}{a} \;\equiv\; m_b + \delta m\,. \quad (2.9)$$

To lowest order in $1/m_b$, the HQET mass formula can now be used to eliminate the unknown expansion parameter, m_b, by expressing it in terms of the physical mass of a b-hadron, specifically the B-meson, and the non-perturbative binding energy, $\mathcal{E}$, which is independent of m_b,

$$M_B = m_b + \mathcal{E} + \mathcal{O}\left(\frac{1}{m_b}\right). \quad (2.10)$$

Using the equation above, we get

$$m_b^{\rm pole} = M_B - \mathcal{E} + \delta m + \mathcal{O}\left(\frac{1}{m_b}\right). \quad (2.11)$$

Finally, the pole mass is converted into the $\overline{MS}$ mass through the well-known one-loop perturbative relation

$$\begin{aligned}\overline{m}_b(\overline{m}_b) \;&=\; m_b^{\rm pole}\left[1\;-\;\frac{4}{3}\left(\frac{\alpha_s(\overline{m}_b)}{\pi}\right)\right]\\ &= \left[M_B - \mathcal{E} + \alpha_s(\overline{m}_b)\,\frac{X_0}{a}\right]\left[1-\frac{4}{3}\left(\frac{\alpha_s(\overline{m}_b)}{\pi}\right)\right] + \mathcal{O}\left(\frac{1}{m_b}\right). \quad (2.12)\end{aligned}$$

We stress that $\overline{m}_b(\overline{m}_b)$ is obtained from the non-perturbative quantity $\mathcal{E}$ combined with the perturbative calculation of lattice ($\alpha_s X_0$) and continuum ($4/3\,\alpha_s(\overline{m}_b)/\pi$) coefficients.

The generalization of eq. (2.12) to higher orders is straightforward. One gets

$$\begin{aligned}\overline{m}_b(\overline{m}_b) &= m_b^{\rm pole}\left[1+\sum_{n=0}^{\infty}\left(\frac{\alpha_s(\overline{m}_b)}{\pi}\right)^{n+1} D_n\right]+\mathcal{O}\left(\frac{1}{m_b}\right)\\ &= \left[M_B-\mathcal{E}+\sum_{n=0}^{\infty}(\alpha_s(\overline{m}_b))^{n+1}\,\frac{X_n}{a}\right]\left[1+\sum_{n=0}^{\infty}\left(\frac{\alpha_s(\overline{m}_b)}{\pi}\right)^{n+1} D_n\right], \end{aligned} \quad (2.13)$$

which is the master equation of our analysis. D_n and X_n are constants which depend on the number of flavours, n_f, the masses of the active quarks and the lattice action used for the light quarks (see below).

The procedure used to calculate $\overline{m}_b(\overline{m}_b)$ is the following [3]:

- compute the binding energy of the B meson in lattice units, $a\mathcal{E}$, using the HQET for the heavy quark and a given discretized action (Wilson, Alpha, Staggered) to describe the light-quark dynamics (we discuss in detail this computation in sections 3 and 4).
- evaluate the value of the lattice spacing, a, from the light-hadron spectroscopy.
- take the experimental value of the B-meson mass, M_B, as input.
- insert the values of these quantities in eq. (2.13) and obtain $\overline{m}_b(\overline{m}_b)$ to a given order in perturbation theory and up to $\mathcal{O}(1/m_b)$ corrections.

A few important remarks are in order at this point:

1. The bare binding energy $\mathcal{E}$ is not a physical quantity since it diverges linearly as $a \to 0$ and needs to be subtracted [19, 20]. Since both the pole and the $\overline{MS}$ mass are finite quantities, instead, the divergence of $\mathcal{E}$ is cancelled by the corresponding divergence of the residual mass δm, expressed in terms of the constants X_n.

2. In practice the cancellation is incomplete because we only know the values of few constants X_n. Therefore, at a given order of the perturbative expansion, we cannot take the lattice spacing too small.

3. The large-n_f approximation shows that the perturbative series for δm, and hence for the pole mass, suffers from renormalon singularities [17, 20]. In other words, the coefficients X_n are expected to grow as const. $\times\, n!$ as $n \to \infty$. These singularities give rise to ambiguities of $\mathcal{O}(\Lambda_{\rm QCD})$ in the sum of the perturbative series. A solution to this problem, which is the one adopted here, is to consider a short distance definition of the b-quark mass, such as the $\overline{MS}$, $\overline{m}_b$, because it is free of renormalon ambiguities (up to $\mathcal{O}(\Lambda^2_{\rm QCD}/m_b)$ at the order at which we are working).

4. In the expression of the $\overline{MS}$ mass, a delicate cancellation of renormalon singularities occurs: the renormalon of the series for δm (with coeffcients X_n) is cancelled by the perturbative expansion of the coefficient relating the pole and the $\overline{MS}$ mass (with coeffcients D_n).

5. In order to achieve the cancellation of renormalon singularities in eq. (2.13), the same coupling constant has to be used in the expansion of δm and in the relation between the pole mass and the $\overline{MS}$ mass [4, 17]. For this reason, although we work at a fixed order of perturbation theory, we believe that the most reasonable choice is to expand both the continuum and lattice series using the same coupling constant.

From the discussion above, it is clear that the precision of our results for the b-quark mass at given $(\mathcal{E},\, a)$ is limited by the number of terms calculated in lattice (X_n) and in continuum (D_n) perturbation theory. The relation between the pole and the $\overline{MS}$ mass has been obtained to $\mathcal{O}(\alpha_s^2)$ by Gray et al. [21] and, recently, to $\mathcal{O}(\alpha_s^3)$ by refs. [22, 23]

$$\begin{aligned} D_0 &= -\frac{4}{3} \\ D_1 &= -11.6656 + 1.0414 \sum_{i=1}^{n_f} \left[1 - \frac{\overline{m}_i}{\overline{m}_b}\right] \\ D_2 &= -157.116 + 23.8779 n_f - 0.6527 n_f^2 \,. \end{aligned} \tag{2.14}$$

Note that the three-loop correction has been evaluated with massless quarks.

As for the residual mass, δm, it can be expressed in terms of the bare lattice coupling, α_0, as

$$\delta m = \sum_{n=0}^{\infty} (\alpha_0)^{n+1} \frac{\overline{X}_n}{a} \,. \tag{2.15}$$

The constant $\overline{X}_0$ is simply $\overline{X}_0 = X_0$ given by the three-dimensional integral

$$X_0 = C_F \, \frac{1}{8\pi^2} \int_{-\pi}^{\pi} d^3k \, \frac{1}{2 \sum_{i=1}^{3} \sin^2(k_i/2)} = 2.1173 \,, \tag{2.16}$$

where $C_F = (N^2-1)/2N$ and N is the number of colors. Martinelli and Sachrajda have performed the calculation of $\overline{X}_1$ by extracting δm from the exponential decrease of the expectation value of large Wilson loops with the perimeter [4]. More recently, Burgio et al. have obtained a preliminary estimate of $\overline{X}_2$ from large Wilson loops computed with the Numerical Stochastic Perturbation Theory (NSPT) on a 24^4 lattice in the quenched approximation [24]. In summary, the results are

$$\begin{aligned} \overline{X}_0 &= 2.1173 \\ \overline{X}_1 &= 11.152 + n_f \, (-\, 0.282 + 0.035 \, c_{SW} - 0.391 \, c_{SW}^2) \\ \overline{X}_2 &= 73.5(9.2) \,, \end{aligned} \tag{2.17}$$

where the value of the coefficient c_{SW} depends on the lattice fermions used in the simulation: for Wilson fermions $c_{SW}=0$, for Clover-SW tree-level improved fermions $c_{SW}=1$ and for the non-perturbatively improved ones, c_{SW} depens on β (see [4] for details). The numerical value of $\overline{X}_2$ has been obtained in the quenched approximation ($n_f=0$) and thus it does not include fermion-loop effects.

The next step is to express δm in terms of the $\overline{MS}$ coupling α_s, i.e. to calculate the coefficients X_n from the $\overline{X}_n$ of eq. (2.17). The relation between α_s and α_0 can be written, to $\mathcal{O}(\alpha_0^3)$, as:

$$\alpha_s(\mu) = \alpha_0 + d_1(\mu a)\,\alpha_0^2 + d_2(\mu a)\,\alpha_0^3 + \cdots \tag{2.18}$$

The pure gauge contributions to d_1 and d_2 have been calculated in ref. [25] and [26] respectively. The quark contribution to d_1 and d_2 for Wilson fermions can be found in [27] and for improved fermions with generic c_{SW} in [4] (see also [28]). So far we have

$$\begin{aligned}
d_1(x) &= -\frac{\beta_0}{2\pi}\ln(x) - \frac{\pi}{2N} + 2.13573N + \\
&\quad + n_f(-0.08413 + 0.0634\,c_{SW} - 0.3750 c_{SW}^2) \\
d_2(x) &= d_1(x)^2 - \frac{17N^2}{12\pi^2}\ln(x) + \frac{3\pi^2}{8N^2} - 2.8626216 + 1.249116\,N^2\,,
\end{aligned} \tag{2.19}$$

where $\beta_0 = \frac{1}{3}(11N - 2n_f)$ and d_2 is given for $n_f=0$. By inverting eq. (2.18) for α_0 and inserting it in eq. (2.15), for $N=3$ we get the values of the constants $X_{0,1,2}$:

$$\begin{aligned}
X_0 &= 2.1173 \\
X_1 &= -1.30533 + 3.70677\ln(\overline{m}_b a) + \\
&\quad + n_f(-0.103872 - 0.224653\ln(\overline{m}_b a) - 0.0992368 c_{SW} + 0.402988 c_{SW}^2) \\
X_2 &= \overline{X}_2 + 6.48945(-3.57877 + \ln(\overline{m}_b a))(3.29596 + \ln(\overline{m}_b a))\,.
\end{aligned} \tag{2.20}$$

Since the value of X_2 is preliminary, we cannot really use it to obtain our final result. We will only show that, even in the quenched case, the result of ref. [24] is not precise enough to obtain a useful information.

3. Lattice computation of $\mathcal{E}$

The bare binding energy of the B meson, $\mathcal{E}$, is measured on the lattice by studying the large-time behaviour of the B meson propagator [29]. The b-quark is described by the discretized version of the HQET,

$$\mathcal{L}_{HQET} = \bar{b}(x) D_4 b(x)\,, \tag{3.1}$$

with the covariant derivative defined as

$$D_\mu b(x) = U_\mu(x) b(x+\hat{\mu}) - b(x)\,, \tag{3.2}$$

JHEP03(2000)018

where $\hat{\mu}$ indicates the μ-direction and $U_\mu(x)$ is the link variable between the lattice sites x and $x+\hat{\mu}$. Light quarks, q, are simulated with some fermion action, in our case the Wilson action.

It is well known that correlation functions involving heavy quarks suffer from a large contamination from higher-mass excitations to the lightest-state contribution, to which we are interested in. In order to improve the isolation of the lightest state, we use cube and double-cube smeared axial-current operators of size L_s, as interpolating operators of the B meson [29]:

$$\begin{aligned} A^L_\mu(x) &= \bar{b}(x)\gamma_\mu\gamma_5 q(x) \\ A^S_\mu(x) &= \sum_i^{L_s} \bar{b}(x_i)\gamma_\mu\gamma_5 q(x) \\ A^D_\mu(x) &= \sum_{i,j}^{L_s} \bar{b}(x_i)\gamma_\mu\gamma_5 q(x_j)\,. \end{aligned} \tag{3.3}$$

From the above operators we construct the two-point correlation functions

$$C^{RR'}_{L_s}(t) = \sum_{\vec{x}} \langle 0|A^R_4(\vec{x},t) A^{R'\dagger}_4(\vec{0},0)|0\rangle\,, \tag{3.4}$$

where R, R' stands for L (Local), S (Single smeared) and D (Double smeared) interpolating operators. The correlation functions are computed after rotating the links in the Coulomb gauge.

At large time distances, $C^{RR'}_{L_s}(t)$ behaves as:

$$C^{RR'}_{L_s}(t) \longrightarrow Z^R_{L_s} Z^{R'}_{L_s} e^{-\mathcal{E}t}\,, \tag{3.5}$$

where

$$Z^R_{L_s} = \frac{1}{\sqrt{2M_B}} \langle 0|A^R_4(\vec{0},0)|B\rangle\,. \tag{3.6}$$

By fitting the large time behaviour of $C^{RR'}(t)$ to eq. (3.5), the bare binding energy can be extracted. Due to contamination from excited states, the interpolating operators couple to the ground state in different ways so that the actual value of $\mathcal{E}$ has some dependence on the cube size and the smearing type used in the analysis. To obtain our best estimate, we compare different methods and account the mixing as a systematic effect contributing to the final error.

We first use the Best Cube Method (BCM): we base our results on the best cube, defined as the one which yields the largest and flattest effective-mass plateau [29]. In practice, we proceed as the following example illustrates. Consider the double smeared interpolating operators. For $t \geq t_{\min}$ and for each L_s, we fit $C^{DD}_{L_s}(t)$ and

$C^{LD}_{L_s}(t)$ to a single-state propagator (3.5). $t_{\rm min}$ is taken as the time at which we start observing a plateau for both the effective mass $\Delta E^{DD}_{L_s}$ and the ratio $\mathrm{R}^{DL/DD}_{L_s}$ (we call BC DL/DD method the case where the smearing is in the sink) or, alternatively, for the effective mass $\Delta E^{LD}_{L_s}$ and the ratio $\mathrm{R}^{LD/DD}_{L_s}$ (we call BC LD/DD method the case where the smearing is in the origin). Effective masses and ratios are defined by

$$\Delta E^{RR'}_{L_s}(t) = \ln\left(\frac{C^{RR'}_{L_s}(t)}{C^{RR'}_{L_s}(t+1)}\right) \to \mathcal{E}$$

$$\mathrm{R}^{RR'/NN'}_{L_s}(t) = \frac{C^{RR'}_{L_s}(t)}{C^{NN'}_{L_s}(t)} \to \frac{Z^{R}_{L_s}\, Z^{R'}_{L_s}}{Z^{N}_{L_s} Z^{N'}_{Ls}}\,. \tag{3.7}$$

For $t \geq t_{\rm min}$, the ground state is assumed to have been isolated. The next step is to combine the exponential fit for $C^{DD}_{L_s}(t)$ ($C^{LD}_{L_s}(t)$) and the average value of the ratio $\mathrm{R}^{DL/DD}_{L_s}(t)$ ($\mathrm{R}^{LD/DD}_{L_s}(t)$) in the plateau region to obtain both $\mathcal{E}$ and Z^L, the matrix element for the local axial current. Similarly, the method is applied to single smeared operators.

A drawback of the BCM is that, in practice, we have only few different cubes at disposal (essentially only two cubes are really useful, $L_s = 7-9$, as suggested by earlier lattice studies). Nevertheless, this method is able to give a reasonable isolation of the lightest state and an accurate determination of the binding energy (although the method is less efficient for the determination of the matrix elements of the local axial current).

In order to improve the accuracy of our analysis, we have also used the multifit method which consists in performing a global fit of the data for all cube sizes and smearing types by minimizing the total χ^2 [29]. Consider, for example, the double-smeared operators. In this case, $\chi^2_{\rm total}$ is defined by

$$\chi^2_{\rm total} = \sum_{t=t_i}^{t_f}\left\{\sum_{L_s=7,9}\left(\frac{C^{LD}_{L_s}(t)_{\rm DATA} - C^{LD}_{L_s}(t)}{\sigma^{LD}_{LS}(t)}\right) + \sum_{L_s=7,9}\left(\frac{C^{DD}_{L_s}(t)_{\rm DATA} - C^{DD}_{L_s}(t)}{\sigma^{DD}_{LS}(t)}\right)^2\right\}, \tag{3.8}$$

where $\sigma^{LD(DD)}_{L_s}$ is the jacknife error of the data points. We also impose the consistency condition that the binding energy $\mathcal{E}$ should be the same for all smearing types and cubes sizes. Moreover, in order to reduce the effect from excited states, it is convenient to fit the data by including the contribution of at least one excited state to $C^{RR'}_{L_s}(t)$.

4. Strategy of the unquenched analysis

Before giving details on the simulation parameters, the calibration of the lattice spacing and the extraction of the binding energy $\mathcal{E}$, we discuss the general strategy followed in the analysis of the unquenched data. This is a crucial issue due to the confusion which, we think, exists in the literature.

JHEP03(2000)018

Our main point, which is justified below, is that the correct procedure consists in performing independent quenched-like calculations of *all* quantities for each fixed value of $k_{\rm sea}$, including the spectroscopy, the calibration of the lattice spacing a and the calculation of the relevant matrix elements [18]. Only when all the quantities are expressed in physical units, the results can be extrapolated in the sea quark mass. Thus, one ends up with a different set of lattice parameters for each $k_{\rm sea}$, such as the critical kappa, k_{cr}, the light quark masses k_u and k_s, the lattice spacing and so on. Extrapolations in the valence quark mass, k_v, should also be performed *at fixed* $k_{\rm sea}$, without ever mixing up the extrapolation in the valence and sea quark masses, which must remain distinct steps of the analysis. In all respects, the value of $k_{\rm sea}$ is an external "field" which controls the link dynamics.

The argument is the following. A change in the value of the sea quark mass(es) modifies the value of the effective coupling constant, because the latter receives contributions from virtual-quark loops. A change of the coupling constant may induce a rapid variation of the value of the lattice spacing which depends exponentially on α_s. Therefore, strictly speaking, lattice results for different $k_{\rm sea}$ correspond to different lattice dynamics and are not directly comparable. Only when the results have been converted to physical units, by using the lattice spacing extracted for each $k_{\rm sea}$, comparisons and extrapolations are possible. We stress again that a combined (in valence and sea quark masses) chiral extrapolation of lattice quantities, as for example the quark masses, may produce incorrect results because in this way we are mixing results corresponding to different values of the lattice spacing and all the parameters of the extrapolation do depend on $k_{\rm sea}$ through a.

Simulation parameters			
$\beta = 5.6$	$V = 24^3 \times 40$	$n_f = 2$	$N_{conf} = 60$
k_v	0.1560, 0.1570, 0.1575, 0.1580		
$k_{\rm sea} = 0.1575$		$k_{\rm sea} = 0.1580$	
$a^{-1} = 2.51(6)$ GeV		$a^{-1} = 2.54(6)$ GeV	
$k_{cr} = 0.15927(5)$		$k_{cr} = 0.15887(4)$	
$k_u = 0.15920(5)$		$k_u = 0.15880(4)$	
$k_s = 0.15747(12)$		$k_s = 0.15715(8)$	

Table 1: Simulation parameters of our run to extract the binding energy $\mathcal{E}$. The values of the lattice spacing, a, and the critical (k_{cr}), light (k_u) and strange (k_s) Wilson parameters are given separately for each sea quark mass (corresponding to $k_{\rm sea}$)(see text).

Having explained our strategy, we turn to the numerical results for the binding energy. We have performed an unquenched computation of $\mathcal{E}$ with two degenerate sea quarks at two values of their mass, $k_{\rm sea} = 0.1575$ and $k_{\rm sea} = 0.1580$. The heavy and light quark propagators have been computed using the set of unquenched link configurations generated by the TχL collaboration. Details of the simulation can be found in ref. [30]. Light quarks are simulated using the Wilson action whereas heavy quarks are described with the discretized HQET. The parameters of our run are given in table 1. The calibration of the lattice spacing has been performed using the K^*-K lattice-plane method of ref. [31].

With the BC method, we find that the flattest and largest plateaus for $\Delta E^{RR'}_{L_s}(t)$ and $\mathrm{R}^{RR'/NN'}_{L_s}(t)$ correspond to the cube size $L_s = 7$ and the correlation functions LD (smearing in the origin) and DD. With the multifit method, we also obtain that a two-state global fit of the correlations LD and DD describes very well the data. The agreement for single smeared interpolating operators is, instead, much worse. Therefore we base our results on double smeared operators since this is the most efficient way of isolating the lightest state. Our best estimates of the values of the binding energy $\mathcal{E}$ are

$$\begin{aligned} a\mathcal{E}_{B_d} &= \{0.588(11)(5), 0.606(15)(2)\} \\ a\mathcal{E}_{B_s} &= \{0.620(8)(5), 0.632(12)(2)\} \end{aligned} \tag{4.1}$$

for the two values of $k_{\rm sea} = 0.1575$ and 0.1580, respectively. The first error is statistical and the second systematic. The latter has been obtained from the spread of our results due to different time intervals of the fit, cube sizes and smearing types. Since a full account of the different methods and evaluation of the uncertainties can be found in ref. [29], we do not give further details here.

5. Sources of systematic error

Using eqs. (2.13), (2.14), (2.20) and (4.1), we can readily obtain the value of the b quark mass. As the value of X_2 is still preliminary and incomplete, we derive our results with the two-loop formula corresponding to the NNLO matching. In order to evaluate the systematic errors on these results, we carefully studied the different sources of uncertainties coming from the use of eq. (2.13) at this order: the value of α_s, higher-order perturbative corrections, input meson mass, method of extracting $\mathcal{E}$, value of $k_{\rm sea}$ and $1/m_b$ corrections. In the following, unless stated otherwise, the central values in the tables correspond to $k_{\rm sea} = 0.1580$, $\mathcal{E}$ has been obtained with the multifit method, the input mass is the B_s meson mass and a linear chiral extrapolation in k_v to k_s has been performed. In the tables, the first error is statistical and the second the systematic one based on the spread of the results due to the uncertainty in the lattice spacing, the different forms of writing eq. (2.13) (see section 5.2) and the use of different time-intervals and fitting methods.

5.1 Dependence of $\overline{m}_b(\overline{m}_b)$ on α_s

In order to obtain $\overline{m}_b(\overline{m}_b)$ we have to choose the value of α_s to be used in the perturbative calculations. At all orders, to cancel the renormalon singularities in eq. (2.13), the same coupling constant has to be used for the lattice and continuum series. For this reason, although our calculation is truncated at $\mathcal{O}(\alpha_s^2)$, we prefer to take the same coupling constant for both the lattice and the continuum cases.

One possibility is to consider the physical value of α_s obtained by running the experimental coupling $\alpha_s(M_Z) = 0.118$ down to $\alpha_s(m_b)$ with $n_f = 5$.

A second possibility is to account that our simulation has been performed with $n_f = 2$ and compute α_s at the NLO with a (still to be determined) $\Lambda_{\rm QCD}^{n_f=2}$. In the quenched case, the value of $\Lambda_{\rm QCD}^{n_f=0} \sim 250\,\mathrm{MeV}$ has been measured in ref. [32] Since the physical value of $\Lambda_{\rm QCD}$ is expected to be larger than the quenched one, in the second case we have used the NLO value of $\alpha_s(m_b)$ obtained by varying $\Lambda_{\rm QCD}^{n_f=2}$ in the range [250, 350] MeV.

Dependence on α_s			
$\Lambda_{\rm QCD}$	n_f	$\alpha_s(\overline{m}_b)$	$\overline{m}_b(\overline{m}_b)$
M_Z	5	0.221	4.31(3)(3)
250	2	0.172	4.25(3)(3)
300	2	0.182	4.26(3)(3)
350	2	0.191	4.27(3)(3)

Table 2: Dependence of $\overline{m}_b(\overline{m}_b)$ on the choice of α_s. For $n_f = 2$ the value of $\Lambda_{\rm QCD}$ is explicitely given; the label M_Z indicates that the physical value of $\alpha_S(M_Z)$ has been used as input to compute $\alpha_s(\overline{m}_b)$ (see text).

The results for different values of α_s are presented in table 2. The dependence on $\Lambda_{\rm QCD}^{n_f=2}$ is very weak: the maximum spread of the values is less than 20 MeV. The difference between the central values obtained with the $n_f = 5$ and $n_f = 2$ coupling is of about 50 MeV. We have taken this as a very conservative estimate of the error due to the choice of the coupling constant.

Note that the value of $\alpha_s(m_b)$ for $\Lambda_{\rm QCD}^{n_f=2} = 300\,\mathrm{MeV}$, $\alpha_s(m_b) = 0.182$, corresponds with a very good approximation to the arithmetic (and geometric) average of the quenched ($n_f = 0$) and physical ($n_f = 5$) couplings ($\alpha_s(m_b) = 0.15$ and $\alpha_s(m_b) = 0.22$ respectively). For this reason our central value for $\overline{m}_b(\overline{m}_b)$ is that computed with $\alpha_s(m_b) = 0.182$.

5.2 Dependence of $\overline{m}_b(\overline{m}_b)$ on higher orders

Eq. (2.13), which is used to evaluate $\overline{m}_b(\overline{m}_b)$ to $\mathcal{O}(\alpha_s^2)$ consists in the product of two factors. We can, then, organize the formula including (not expanding) or excluding (expanding to $\mathcal{O}(\alpha_s^2)$) the $\mathcal{O}(\alpha_s^3)$ terms arising from the product. We take the difference between these (formally equivalent) procedures, as an estimate of unknown higher-order terms in perturbation theory (PT). In table 3, the values of $\overline{m}_b(\overline{m}_b)$ obtained from the expanded and not expanded forms of eq. (2.13) are presented. For $n_f = 2$ the dependence of our results on higher orders is $30 \div 40\,\mathrm{MeV}$, for $n_f = 5$ the difference is 60 MeV. From these spreads, we conclude that a fair estimate of the effect of expanding or not expanding eq. (2.13) is $\sim 50\,\mathrm{MeV}$. From this estimate we assume from higher-order terms an error of $\pm 25\,\mathrm{MeV}$.

As best estimate of $\overline{m}_b(\overline{m}_b)$, for each choice of the value of the coupling constant, we take the average of the results obtained with the not-expanded and expanded form of the master formula. In this way we have computed the central values given in tables 2, 4, 5, 6 and 7.

In order to get an independent estimate of the systematic uncertainty due to higher-order terms in the perturbative espansion, we also tried to compute $\overline{m}_b(\overline{m}_b)$ using for $\overline{X}_2$ the preliminary result of eq. (2.20). In the numerical calculations, obtained with the expanded form of eq. (2.13), we allowed $\overline{X}_2$ to vary in the 1σ

Dependence on higher orders (I)				
$\Lambda_{\rm QCD}$	n_f	$\alpha_s(\overline{m}_b)$	$\overline{m}_b(\overline{m}_b)$ *not expanded*	$\overline{m}_b(\overline{m}_b)$ *expanded*
M_Z	5	0.221	4.28(3)(1)	4.34(3)(1)
300	2	0.182	4.24(3)(1)	4.28(3)(1)
Dependence on higher orders (II)				
$\Lambda_{\rm QCD}$	n_f	$\alpha_s(\overline{m}_b)$	$\overline{m}_b(\overline{m}_b)$ at $\mathcal{O}(\alpha_s^2)$	$\overline{m}_b(\overline{m}_b)$ at $\mathcal{O}(\alpha_s^3)$
M_Z	5	0.221	4.34(3)(1)	$4.16 \div 4.34$
300	2	0.182	4.28(3)(1)	$4.10 \div 4.27$

Table 3: Dependence of $\overline{m}_b(\overline{m}_b)$ on higher orders in PT: (I) expanded and not expanded form of the master equation; (II) excluding or including preliminary (incomplete) NNLO contributions. For $n_f = 2$ the value of $\Lambda_{\rm QCD}$ is explicitely given; the label M_Z indicates that the physical value of $\alpha_S(M_Z)$ has been used as input to compute $\alpha_s(\overline{m}_b)$.

Dependence on M_B				
$\Lambda_{\rm QCD}$	n_f	$\alpha_s(\overline{m}_b)$	$\overline{m}_b(\overline{m}_b)$ from B_d	$\overline{m}_b(\overline{m}_b)$ from B_s
M_Z	5	0.221	4.28(4)(3)	4.31(3)(3)
300	2	0.182	4.23(4)(3)	4.26(3)(3)

Table 4: Dependence of $\overline{m}_b(\overline{m}_b)$ on the input B mesons masses (see text).

interval [64.3, 82.7], obtaining the range of masses given in table 3 (only the central values are given). The values of $\overline{m}_b(\overline{m}_b)$ at order $\mathcal{O}(\alpha_s^2)$ are also given for comparison. There are huge numerical cancellations occurring in the calculation of X_2 from $\overline{X}_2$ in eq. (2.20). For this reason, the difference between the NNLO and the (approximate) $\mathrm{N^3LO}$ results varies from about zero to 180 MeV, depending on the value of $\overline{X}_2$. This quantity, even in the quenched case, is still affected by such a large uncertainty [24] that it is impossible to use it for any realistic estimate. We urgently call for a more precise determination of $\overline{X}_2$ in both the unquenched and quenched cases.

5.3 Dependence of $\overline{m}_b(\overline{m}_b)$ on the input B meson mass

Consistent values of $\overline{m}_b(\overline{m}_b)$ should be obtained using as input either the B_d or the B_s meson masses. The corresponding values of the binding energy, $\mathcal{E}_{B_d}$ and $\mathcal{E}_{B_s}$, respectively, are used in the two cases. This checks the lattice value of the M_{B_d}-M_{B_s} mass splitting and the smoothness of our chiral extrapolation. Actually, for the B_s meson, the physical value of the strange quark mass, corresponding to k_s, is within the range of valence quark masses (see table 1) and only a mild interpolation, rather than an extrapolation, is needed. For the B_d meson, instead, we have extrapolated almost to the chiral limit. In order to compute the pole mass we have taken $M_{B_d} = 5.279$ and $M_{B_s} = 5.375$ GeV [33]. In table 4, we compare our results for the two cases. The results are nicely compatible and a small difference ($\simeq 30$ MeV) is observed in the two cases.

Dependence on the method of extracting $\mathcal{E}$				
$\Lambda_{\rm QCD}$	n_f	$\alpha_s(\overline{m}_b)$	LD/DD with $L_s=7$	Multifit
M_Z	5	0.221	4.34(2)(4)	4.31(3)(3)
300	2	0.182	4.29(2)(3)	4.26(3)(3)

Table 5: Dependence of $\overline{m}_b(\overline{m}_b)$ on the method to extract $\mathcal{E}$ (see text).

Dependence on $k_{\rm sea}$				
$\Lambda_{\rm QCD}$	n_f	$\alpha_s(\overline{m}_b)$	$\overline{m}_b(\overline{m}_b)$ with $k_{\rm sea}=0.1575$	$\overline{m}_b(\overline{m}_b)$ with $k_{\rm sea}=0.1580$
$M_Z/$	5	0.221	4.34(2)(4)	4.31(3)(3)
300	2	0.182	4.29(2)(4)	4.26(3)(3)

Table 6: Dependence of $\overline{m}_b(\overline{m}_b)$ on $k_{\rm sea}$(see text).

5.4 Dependence of $\overline{m}_b(\overline{m}_b)$ on the method for extracting $\mathcal{E}$

In section 3, we discussed the two methods used to determine the binding energy $\mathcal{E}$: the Best Cube method and the Multifit method. If the lightest state has been well isolated, both methods should give compatible results for the b quark mass. In table 5, we present the value of $\overline{m}_b(\overline{m}_b)$ for the BC method (obtained with LD/DD, smearing in the origin, and $L_s=7$, which gives the flattest and largest plateau) and for the multifit method. Also in this case the results differ by $\simeq 30\,\mathrm{MeV}$.

5.5 Dependence of $\overline{m}_b(\overline{m}_b)$ on $k_{\rm sea}$

As discussed before, in order to compare the values obtained for different $k_{\rm sea}$ and attempt an extrapolation, we have first to convert the lattice quantities to physical units. In table 6, the values of $\overline{m}_b(\overline{m}_b)$ for either values of $k_{\rm sea}$ are given. The dependence of our results on $k_{\rm sea}$, in the sea-quark mass region of our simulation, is small. Indeed, taking into account that they correspond to independent simulations, we are not able to observe any dependence on $k_{\rm sea}$ within errors. Therefore, the only possible strategy is not to attempt an extrapolation in $k_{\rm sea}$ and take the value at the lightest $k_{\rm sea}$, i.e. $k_{\rm sea}=0.1580$, as the best estimate of the physical value of the b quark mass. The difference between the results at the two values of $k_{\rm sea}$ is accounted as a systematic effect in the final error.

5.6 $1/m_b$ correcctions to $\overline{m}_b(\overline{m}_b)$

Our whole analysis is performed to the lowest order of the expansion in $1/m_b$. This means that $\mathcal{O}(1/m_b)$ contributions to the relation between the QCD and the lattice HQET quantities in eqs. (2.1) and (2.10) have been neglected. We now make an estimate of the error introduced by these higher-order corrections.

JHEP03(2000)018

The HQET pseudoscalar mass formula including $1/m_b$ corrections is given by

$$M_B = m_b + \mathcal{E} - \frac{\lambda_1}{2\,m_b} - \frac{3\,\lambda_2}{2\,m_b} + \mathcal{O}\left(\frac{1}{m_b^2}\right), \tag{5.1}$$

where the parameters λ_1 and λ_2 are matrix elements between B states of the kinetic and chromomagnetic operators

$$\lambda_1 \equiv \frac{\langle B|\,\bar{b}\,\vec{D}^2\,b\,|B\rangle}{2\,M_B}$$
$$\lambda_2 \equiv \frac{\langle B|\,\bar{b}\,(\vec{S}\cdot g\vec{B})\,b\,|B\rangle}{M_B}, \tag{5.2}$$

with $\vec{S}$ the spin operator of the B-meson and $\vec{B}$ the chromomagnetic field ($B_i = \frac{1}{2}\,\epsilon_{ijk}\,G_{jk}$).

It is straightforward to estimate λ_2 because which is related to the vector-pseudoscalar mass splitting,

$$\lambda_2 = \frac{1}{4}\left(M_{B^*}^2 - M_B^2\right) \approx 0.12\,\mathrm{GeV}^2\,. \tag{5.3}$$

The extraction of λ_1, is, instead, more difficult as demonstrated by the spread of values obtained with different approaches: the lepton-energy spectrum in inclusive semileptonic B decays using Zero Recoil sum rules, QCD sum rules, experiment data analysis and the HQET Virial theorem (see [34] and references therein). It has also been estimated on the lattice using the discretized HQET [3, 35]. Although biased by the lattice results, we prefer a small value for this parameter, in the absence of an accurate determination we let it to vary in the interval $-0.5 \div 0.0\,\mathrm{GeV}^2$. With this range, we find that the contribution of the $1/m_b$ corrections to the pole mass is at most $\simeq 30\,\mathrm{MeV}$. Due to the theoretical uncertainties on λ_1, we do not attempt to correct the $1/m_b$ terms but include their effect as a sytematic error on the final result.

5.7 Continuum limit of $\overline{m}_b(\overline{m}_b)$

To date, in quenched lattice simulations the binding energy $a\mathcal{E}$ has been computed at three values of $\beta = 6.0$, 6.2 and 6.4 [29].[1] In table 7 we give the values of the b-quark mass from these quenched simulations for different lattice spacings. The quenched results are computed using the values of the binding energy from the

Dependence on a in the quenched case

β	a^{-1} (GeV)	$a\mathcal{E}_{B_d}$	$\overline{m}_b(\overline{m}_b)$
6.0	2.0(2)	0.61(1)	4.34(5)(10)
6.2	2.9(3)	0.52(1)	4.29(7)(10)
6.4	3.8(3)	0.460(7)	4.25(7)(10)

Table 7: Results for $\overline{m}_b(\overline{m}_b)$ at different values of the lattice spacing a. The first error is due to the uncertainties in the lattice determination of the binding energy, and the second is an estimate of the higher order perturbative corrections.

[1] Finite volume effects may be present in the results at $\beta = 6.4$ since the lattice volume was rather small.

APE collaboration [29] and NNLO quenched master formula with the coupling constant $\alpha_s(\overline{m}_b) = 0.15$. The quenched values are very close to our new result with $n_f = 2$. Although one may argue that there is a (rather mild) tendency towards lower values as a decreases, with the present uncertainties we cannot attempt any extrapolation in a or realistic estimate of the discretization errors.

6. Final result for $\overline{m}_b(\overline{m}_b)$ and comparison with other determinations

We consider as best estimate of $\overline{m}_b(\overline{m}_b)$ the value obtained with the pole mass extracted by using the mass of the B_s meson and the binding energy $\mathcal{E}_{B_s}$ measured on the lattice through the multifit method at $k_{\text{sea}} = 0.1580$, by averaging the results of the expanded and not expanded form of eq. (2.13) and by taking the coupling constant α_s computed at NLO with $n_f = 2$ and $\Lambda_{\text{QCD}}^{n_f=2} = 300\,\text{MeV}$. Using the estimate of the different errors discussed in the previous section we then obtain:

$$\overline{m}_b(\overline{m}_b) = (4.26 \pm 0.03\ \pm 0.05 \pm 0.07)\,\text{GeV}\,. \tag{6.1}$$

The first error is statistical. The second is the systematic error obtained from the spread of values due to the use of different time intervals, fitting methods, smearing types and cube sizes for the interpolating operators, the dependence of the results on the k_{sea}, the calibration of the lattice spacing and the estimate of the $1/m_b$ corrections. Finally, the third is an estimate of the error due to the uncertainties in the values of α_s and to the effects of higher-order terms in eq. (2.13). We find that the latter is the most important source of error in the final result. For this reason a big effort must be done to compute the unknown N^3LO contributions to the residual mass on the lattice and the NNLO matching coefficient between the lattice and continuum α_s in the unquenched case. On the numerical side, a non-perturbative calculation of $\Lambda_{\text{QCD}}^{n_f=2}$ is also important.

Our new result (6.1) modifies and improve the previous one obtained from quenched lattice simulations with NLO matching only [3]

$$\overline{m}_b(\overline{m}_b) = (4.15 \pm 0.05\ \pm 0.20)\,\text{GeV}\,, \tag{6.2}$$

where the first error is due to the lattice systematics and the second is an estimate of higher orders.

It is interesting to compare eq. (6.1) with recent values obtained with completely different approaches as mass effects in 3-jets $b\bar{b}g$ events, $b\bar{b}$ production cross-section and Υ spectroscopy. Our final result is in good agreement with most of NNLO estimates, as shown in figure 1. In the figure we also give our world average and error. This average has been obtained by using only the most recent NNLO determinations from Υ spectroscopy and lattice QCD, i.e. we did not use the results of refs. [3, 5, 8, 9],

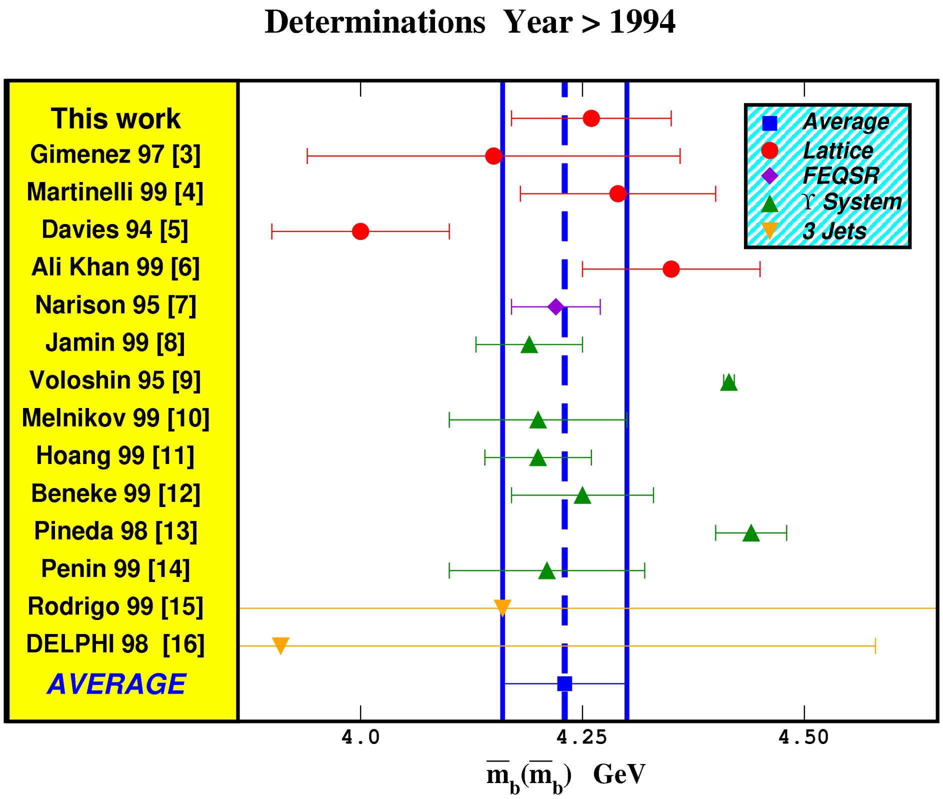

Figure 1: Determinations of $\overline{m}_b(\overline{m}_b)$ from different methods compared with our result. Only references after 1994 have been included.

either because they have been superseeded by more accurate calculations or because they are only computed at the NLO accuracy. The average is

$$\overline{m}_b(\overline{m}_b) = 4.23 \pm 0.07\,\mathrm{GeV}\,, \tag{6.3}$$

which corresponds to a relative error of less than 2% comparable to the precision on the top quark mass. The masses of the quarks of the heaviest and last discovered generation are, and will probably remain, the most accurately determined quark masses.

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References

[1] C.R. Allton et al., *Quark masses from lattice QCD at the next-to-leading order*, *Nucl. Phys.* **B 431** (1994) 667 [hep-ph/9406263];
R. Gupta and T. Bhattacharya, *Light quark masses from lattice QCD*, *Phys. Rev.* **D 55** (1997) 7203 [hep-lat/9605039];
B.J. Gough et al., *The light quark masses from lattice gauge theory*, *Phys. Rev. Lett.* **79** (1997) 1622 [hep-ph/9610223];
C.R. Allton, V. Gimenez, L. Giusti and F. Rapuano, *Light quenched hadron spectrum and decay constants on different lattices*, *Nucl. Phys.* **B 489** (1997) 427 [hep-lat/9611021];
SESAM collaboration, *Light and strange hadron spectroscopy with dynamical Wilson fermions*, *Phys. Rev.* **D 59** (1999) 014509 [hep-lat/9806027];
A. Cucchieri, M. Masetti, T. Mendes and R. Petronzio, *Non-perturbatively improved quenched hadron spectroscopy*, *Phys. Lett.* **B 422** (1998) 212 [hep-lat/9711040];
M. Gockeler et al., *Scaling of non-perturbatively O(a) improved wilson fermions: hadron spectrum, quark masses and decay constants*, *Phys. Rev.* **D 57** (1998) 5562 [hep-lat/9707021];
V. Gimenez, L. Giusti, F. Rapuano and M. Talevi, *Lattice quark masses: a non-perturbative measure*, *Nucl. Phys.* **B 540** (1999) 472 [hep-lat/9801028];
D. Becirevic et al., *Non-perturbatively renormalized light-quark masses with the ALPHA action*, *Phys. Lett.* **B 444** (1998) 401 [hep-lat/9807046];
ALPHA collaboration, J. Garden, J. Heitger, R. Sommer and H. Wittig, *Precision computation of the strange quark's mass in quenched QCD*, hep-lat/9906013;
M. Gockeler et al., *A lattice determination of light quark masses*, hep-lat/9908005;
JLQCD collaboration, *Non-perturbative determination of quark masses in quenched lattice QCD with the Kogut-Susskind fermion action*, *Phys. Rev. Lett.* **82** (1999) 4392 [hep-lat/9901019];
CP-PACS collaboration, *Quenched light hadron spectrum*, *Phys. Rev. Lett.* **84** (2000) 238 [hep-lat/9904012];
T. Blum, A. Soni and M. Wingate, *Calculation of the strange quark mass using domain wall fermions*, *Phys. Rev.* **D 60** (1999) 114507 [hep-lat/9902016];
D. Becirevic, V. Gimenez, V. Lubicz and G. Martinelli, *Light quark masses from lattice quark propagators at large momenta*, hep-lat/9909082.

[2] A. Pich and J. Prades, *Strange quark mass determination from Cabibbo-suppressed tau decays*, *J. High Energy Phys.* **10** (1999) 004 [hep-ph/9909244];
A. Pich and J. Prades, *Tau-decay determination of the strange quark mass*, hep-ph/9909559;
S. Narison, *On the strange quark mass from e^+e^- and tau decay data and test of the* SU(2) *isospin symmetry*, *Phys. Lett.* **B 466** (1999) 345 [hep-ph/9905264];
ALEPH collaboration, *Study of τ decays involving kaons, spectral functions and determination of the strange quark mass*, *Eur. Phys. J.* **C 11** (1999) 599 [hep-ex/9903015].

[3] M. Crisafulli, V. Gimenez, G. Martinelli and C.T. Sachrajda, *First lattice calcula-*

tion of the B-meson binding and kinetic energies, *Nucl. Phys.* **B 457** (1995) 594 [hep-ph/9506210];
V. Gimenez, G. Martinelli and C.T. Sachrajda, *A high statistics lattice calculation of the B meson binding energy*, *Phys. Lett.* **B 393** (1997) 124 [hep-lat/9607018].

[4] G. Martinelli and C.T. Sachrajda, *Computation of the B-quark mass with perturbative matching at the next-to-next-to-leading order*, *Nucl. Phys.* **B 559** (1999) 429 [hep-lat/9812001].

[5] C.T.H. Davies et al., *A new determination of M_b using lattice QCD*, *Phys. Rev. Lett.* **73** (1994) 2654 [hep-lat/9404012].

[6] A.A. Khan et al., *Heavy-light mesons and baryons with b quarks*, hep-lat/9912034.

[7] S. Narison, *How reliable are the HQET sum rule predictions?*, *Phys. Lett.* **B 352** (1995) 122 [hep-ph/9501402].

[8] M. Jamin and A. Pich, *Bottom quark mass and α_s from the Υ system*, *Nucl. Phys.* **B 507** (1997) 334 [hep-ph/9702276]; *Bottom quark mass from QCD sum rules for the upsilon system*, *Nucl. Phys.* **74** *(Proc. Suppl.)* (1999) 300 [hep-ph/9810259].

[9] M.B. Voloshin, *Precision determination of α_s and m_b from QCD sum rules for $b\bar{b}$*, *Int. J. Mod. Phys.* **A 10** (1995) 2865 [hep-ph/9502224].

[10] K. Melnikov and A. Yelkhovsky, *The b quark low-scale running mass from Υ sum rules*, *Phys. Rev.* **D 59** (1999) 114009 [hep-ph/9805270].

[11] A.H. Hoang, *Heavy quark masses from the $Q\bar{Q}$ threshold and the Υ expansion*, hep-ph/9909356; *1S and $\overline{MS}$ bottom quark masses from Υ sum rules*, *Phys. Rev.* **D 61** (2000) 034005 [hep-ph/9905550];
A.H. Hoang, *Bottom quark mass from Υ mesons*, *Phys. Rev.* **D 59** (1999) 014039 [hep-ph/9803454];
A.H. Hoang and A.V. Manohar, *Charm effects in the $\overline{MS}$ bottom quark mass from Υ mesons*, hep-ph/9911461.

[12] M. Beneke and A. Signer, *The bottom $\overline{MS}$ quark mass from sum rules at next-to-next-to-leading order*, *Phys. Lett.* **B 471** (1999) 233 [hep-ph/9906475];
M. Beneke, A. Signer and V.A. Smirnov, *A two-loop application of the threshold expansion: the bottom quark mass from $b\bar{b}$ production*, hep-ph/9906476.

[13] A. Pineda and F.J. Yndurain, *Calculation of quarkonium spectrum and m_b, m_c to order α_s^4*, *Phys. Rev.* **D 58** (1998) 094022 [hep-ph/9711287];
Comment on "Calculation of quarkonium spectrum and m_b, m_c to order α_s^4", *Phys. Rev.* **D 61** (2000) 077505 [hep-ph/9812371].

[14] A.A. Penin and A.A. Pivovarov, *Next-to-next-to-leading order vacuum polarization function of heavy quark near threshold and sum rules for $b\bar{b}$ system*, *Phys. Lett.* **B 435** (1998) 413 [hep-ph/9803363]; *Bottom quark pole mass and $|V_{cb}|$ matrix element from*

$R(e^+e^- \to b\bar{b})$ and $\Gamma_{sl}(b \to cl\nu_l)$ in the next-to-next-to-leading order, *Nucl. Phys.* **B 549** (1999) 217 [hep-ph/9807421].

[15] G. Rodrigo, M. Bilenky and A. Santamaria, *Quark-mass effects for jet production in e^+e^- collisions at the next-to-leading order: results and applications*, *Nucl. Phys.* **B 554** (1999) 257 [hep-ph/9905276];
G. Rodrigo, A. Santamaria and M. Bilenkii, *Do the quark masses run? extracting $m_b(m_Z)$ from LEP data*, *Phys. Rev. Lett.* **79** (1997) 193 [hep-ph/9703358];
M. Bilenkii et al., *$m_b(m_Z)$ from jet production at the Z peak in the Cambridge algorithm*, *Phys. Rev.* **D 60** (1999) 114006 [hep-ph/9807489].

[16] DELPHI collaboration, *m_b at m_Z*, *Phys. Lett.* **B 418** (1998) 430.

[17] G. Martinelli and C.T. Sachrajda, *On the difficulty of computing higher-twist corrections*, *Nucl. Phys.* **B 478** (1996) 660 [hep-ph/9605336].

[18] V. Gimenez, L. Giusti, F. Rapuano and G. Martinelli, *NNLO unquenched calculation of the b quark mass*, hep-lat/9909138.

[19] L. Maiani, G. Martinelli and C.T. Sachrajda, *Nonperturbative subtractions in the heavy quark effective field theory*, *Nucl. Phys.* **B 368** (1992) 281.

[20] M. Beneke and V.M. Braun, *Heavy quark effective theory beyond perturbation theory: renormalons, the pole mass and the residual mass term*, *Nucl. Phys.* **B 426** (1994) 301 [hep-ph/9402364];
I.I. Bigi, M.A. Shifman, N.G. Uraltsev and A.I. Vainshtein, *The pole mass of the heavy quark. perturbation theory and beyond*, *Phys. Rev.* **D 50** (1994) 2234 [hep-ph/9402360];
M. Beneke, *Renormalons*, *Phys. Rep.* **317** (1999) 1 [hep-ph/9807443].

[21] N. Gray, D. J. Broadhurst, W. Grafe and K. Schilcher, *Three loop relation of quark (modified) MS and pole masses*, *Z. Physik* **C 48** (1990) 673.

[22] K.G. Chetyrkin and M. Steinhauser, *Short distance mass of a heavy quark at order α_s^3*, *Phys. Rev. Lett.* **83** (1999) 4001 [hep-ph/9907509];
K.G. Chetyrkin and A. Retey, *Renormalization and running of quark mass and field in the regularization invariant and $\overline{MS}$ schemes at three and four loops*, hep-ph/9910332.

[23] K. Melnikov and T. van Ritbergen, *The three-loop relation between the $\overline{MS}$ and the pole quark masses*, hep-ph/9912391.

[24] G. Burgio, F.D. Renzo, M. Pepe and L. Scorzato, *β-function, renormalons and the mass term from perturbative wilson loops*, hep-lat/9909169.

[25] A. Hasenfratz and P. Hasenfratz, *The connection between the Λ parameters of lattice and continuum QCD*, *Phys. Lett.* **B 93** (1980) 165.

[26] M. Lüscher and P. Weisz, *Two loop relation between the bare lattice coupling and the MS coupling in pure* SU(n) *gauge theories*, *Phys. Lett.* **B 349** (1995) 165 [hep-lat/9502001].

[27] H. Kawai, R. Nakayama and K. Seo, *Comparison of the lattice* Λ *parameter with the continuum* Λ *parameter in massless QCD*, *Nucl. Phys.* **B 189** (1981) 40;
C. Christou, A. Feo, H. Panagopoulos and E. Vicari, *The three-loop* β*-function of* SU(N) *lattice gauge theories with Wilson fermions*, *Nucl. Phys.* **B 525** (1998) 387 [hep-lat/9801007].

[28] S. Sint and R. Sommer, *The running coupling from the QCD Schrödinger functional: a one loop analysis*, *Nucl. Phys.* **B 465** (1996) 71.

[29] APE collaboration, *A high statistics lattice calculation of* f_B *in the static limit on APE*, *Nucl. Phys.* **B 413** (1994) 461; *A high statistics lattice calculation of* $f_B^{\rm static}$ *at* $\beta = 6.2$ *using the clover action*, *Phys. Lett.* **B 326** (1994) 295 hep-lat/9402343;
APE collaboration, *Results for the B meson decay constant from the APE collaboration*, *Nucl. Phys.* **42** *(Proc. Suppl.)* (1995) 385 [hep-lat/9502013].

[30] T. Lippert et al., *Sesam and Tchil results for Wilson action: a status report*, *Nucl. Phys.* **60A** *(Proc. Suppl.)* (1998) 311 [hep-lat/9707004].

[31] C.R. Allton, V. Gimenez, L. Giusti and F. Rapuano, *Light quenched hadron spectrum and decay constants on different lattices*, *Nucl. Phys.* **B 489** (1997) 427 [hep-lat/9611021].

[32] M. Lüscher, R. Sommer, U. Wolff and P. Weisz, *Computation of the running coupling in the* SU(2) *Yang-Mills theory*, *Nucl. Phys.* **B 389** (1993) 247 [hep-lat/9207010]; *A precise determination of the running coupling in the* SU(3) *Yang-Mills theory*, *Nucl. Phys.* **B 413** (1994) 481 [hep-lat/9309005].

[33] Particle Data Group, *Review of particle physics*, *Eur. Phys. J.* **C 3** (1998) 1.

[34] G. Martinelli, *Theoretical review of B physics*, *Nucl. Instrum. Meth.* **A384** (1996) 241 [hep-ph/9610455].

[35] V. Gimenez, G. Martinelli and C.T. Sachrajda, *A high-statistics lattice calculation of* λ_1 *and* λ_2 *in the B meson*, *Nucl. Phys.* **B 486** (1997) 227 [hep-lat/9607055]; *B physics on the lattice:* $\bar\lambda$, λ_1, $\overline{m}_b(\overline{m}_b)$, λ_2, B^0-$\bar B^0$ *mixing*, f_B *and all that*, *Nucl. Phys.* **53** *(Proc. Suppl.)* (1997) 365 [hep-lat/9608039].